Initial-Value Problem for the Non-local Generalization of the Lorentz-Dirac Equation

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Although the equation of motion, recently proposed for the classical radiating electron, is of non-local character in proper time, the Newtonian initial data (position and velocity) are sufficient to guarantee existence and uniqueness of the solutions. The corresponding existence proof is accomplished by the Picard-Lindelöf method of successive approximations. This method indicates the possibility of a perturbation expansion of the exact solution in terms of the non-locality parameter. Such a perturbation expansion does not seem to be possible in the Lorentz-Dirac theory.

I. Introduction and Survey of Results

It is a well-known fact in classical point mechanics that the temporal development of the motion of a point particle is uniquely determined, if its initial position and velocity are specified. The impact of special relativity does not modify anything in this statement referring to a quite general class of second order differential equations.

A certain problem in connection with the initial data arises not until one wants to include radiation reaction into the equation of motion in the case of charged particles. The theory, which presently seems largely accepted in this field, is the Lorentz-Dirac theory [1], mainly promoted by Rohrlich [2] (for the present context see p. 52 of the latter reference). But the basic equation of motion *

$$m\,c^2\,\dot w_{(s)} = K_{(s)} + \tfrac{2}{3}\,Z^2\,\ddot w_{(s)} \qquad (\mathrm{I},1)$$

of this theory is of second order in "velocity" $w_{(s)}$ and therefore the initial "acceleration" $\dot w$ has to be specified in addition to position and velocity.

Unfortunately, one cannot prescribe an arbitrary initial acceleration, if one does not want to have an ever increasing velocity (runaway solution) even if the particle has escaped from a domain of interaction of finite range. In order to avoid difficulties of this sort, Rohrlich [2] has proposed to convert the differential Eq. (I, 1) in an integro-differential equation

$$m\,c^2\,\dot w_{(s)} = \int_{s'=s}^{\infty} \exp\{-(s'-s)/\Delta s\}\, K_{(s')}\,\mathrm{d}s'/\Delta s\,;$$
$$\Delta s = \tfrac{2}{3}\,Z^2/m\,c^2\,. \qquad (\mathrm{I},2)$$

If the force $K_{(s)}$ is a known function of proper time s, the initial acceleration (at $s=0$, say, without loss of generality) follows uniquely from this equation. On the other hand, the unphysical pre-acceleration phenomena are fully elucidated in this form (I, 2) of the equation of motion and provide reason enough to be unsatisfied with this theory.

In order to get rid of all these difficulties, a non-local generalization of the Lorentz-Dirac theory has recently been proposed [3, 4], which actually is able to avoid runaway solutions and also pre-acceleration in a certain sense. But since this new theory is non-local in proper time s:

$$m\,c^2\,\dot w_{(s)}\,\mathrm{Cosh}\,\{w_{(s+\Delta s)} - w_{(s)}\} = K_{(s)}\,, \quad (\mathrm{I},3)$$

and thus corresponds to an infinitely high number of derivatives, one might suppose, that one has to specify an infinitely high number of initial data (referring to all the higher derivatives) or, equivalently, to specify the "velocity" $w_{(s)}$ in an initial interval of length Δs. If the latter would have been done, one could easily find the corresponding solution $w_{(s)}$ interval by interval from resolving (I, 3) as

$$w_{(s+\Delta s)} = w_{(s)} + \mathrm{Ar\,Cosh}\,K_{(s)}/m\,c^2\,\dot w_{(s)}\,. \quad (\mathrm{I},4)$$

Indeed, if one consults a standard mathematical text on this subject [5], one finds existence and uniqueness theorems referring to this kind of initial conditions.

But in the present paper we shall show that the Newtonian initial data (namely position and velocity) are sufficient to determine uniquely the future motion of the extended electron described by (I, 3).

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* In this paper we restrict ourselves to one-dimensional motion, represented by the four-velocity $\{u^\lambda\} = \{\mathrm{d}z^\lambda/\mathrm{d}s\} = \{\mathrm{Cosh}\,w;\,0,0,\,\mathrm{Sinh}\,w\}$. Moreover, we concentrate on the determination of "velocity" $w_{(s)}$ in terms of its initial value $w_{\mathrm{in}} = w_{(0)}$. The spatial motion is then obtainable from here by a further integration.

So our finite-size model restores the old simplicity of initial conditions, which had to be abandoned in the rather doubtful Lorentz-Dirac theory. The only restriction, which becomes necessary in the course of the following existence proof, consists in the limitation of the force $K_{(s)}$ not to surpass in magnitude the interaction force of two Coulomb singularities at distance of two times the classical electron radius. Clearly, such strong forces are far beyond experimental relevance; and the existence and uniqueness of solutions in physically reasonable situations is guaranteed.

Nevertheless, the restriction mentioned seems to be not mandatory, because we are able to find an exact solution for very short lasting pulses, which may violate the restriction under consideration. So one cannot exclude that there exists a proof working without this restriction.

The proof given here is accomplished by making use of the Picard-Lindelöf method of successive approximations. Applying this procedure to the constant-force problem, it is suggested that the exact solution of the non-local equation of motion is an analytic function of the size parameter Δs. This would mean that the Lorentz-Dirac solution (I, 2), being non-analytic in Δs in the general case, can by no means be regarded as an approximation for the finite-size theory. Presumably, there does not exist a finite-size theory at all, which has the Lorentz-Dirac theory as a consistent point limit.

II. Existence Proof by Means of the Picard-Lindelöf Method

In order to prove that there exists at least one solution of the equation of motion (I, 3), which satisfies the Newtonian initial conditions, we resort to the famous Picard-Lindelöf method evoked in mathematical literature [6] to perform the analogous proof for differential equations of the kind

$$\dot{w}_{(s)} \equiv \mathrm{d}w_{(s)}/\mathrm{d}s = \mathfrak{f}(s, w) . \tag{II, 1}$$

For this case, the Picard-Lindelöf existence theorem says that there exists exactly one solution of (II, 1), which is continuous together with its first derivative in a certain region $\mathfrak{G}(s, w)$ and passes through a given point $(s^*, w^*) \in \mathfrak{G}$, if $\mathfrak{f}(s, w)$ is itself continuous in $\mathfrak{G}$ and satisfies there the Lipschitz condition

$$|\mathfrak{f}(s, w_2) - \mathfrak{f}(s, w_1)| \leqq A\,|w_2 - w_1| \tag{II, 2a}$$

$$|\mathfrak{f}(s, w)| \leqq B\,, \quad A, B| = \text{const.} \tag{II, 2b}$$

But writing the equation of motion (I, 3) in the form [abbreviate $F_{(s)} := K_{(s)}/m\,c^2$]:

$$\dot{w}_{(s)} = \frac{F_{(s)}}{\operatorname{Cosh}[w_{(s+\Delta s)} - w_{(s)}]}\,, \tag{II, 3}$$

we recognize that we have now to deal with an equation of the kind

$$\dot{w}_{(s)} = \mathfrak{f}(s; \Delta w_{(s)})\,; \quad \Delta w_{(s)} = w_{(s+\Delta s)} - w_{(s)} \tag{II, 4}$$

or more generally

$$\dot{w}_{(s)} = \mathfrak{f}(s; w_{(s)}, w_{(s+\Delta s)}) \tag{II, 4a}$$

instead of (II, 1). Clearly, in the present case

$$\mathfrak{f}(s; w_{(s)}, w_{(s+\Delta s)}) = \mathfrak{f}(s, \Delta w) = F_{(s)}/\operatorname{Cosh}\Delta w\,. \tag{II, 5}$$

It is not necessary for the following existence proof to refer to the special case (II, 5). Rather, we deal with the general case (II, 4a).

According to the principal idea of the method of successive approximations ** one has to construct a sequence of functions $w_{k(s)}$, which have to be shown to converge uniformly on a limit function $W_{(s)}$

$$\lim_{k\to\infty} w_{k(s)} = W_{(s)}\,, \tag{II, 6}$$

so that $W_{(s)}$ satisfies Eq. (II, 4a) and has $W_{(s=0)} = w_{\text{in}}$. Clearly, we choose for the first approximation $w_{0(s)} \equiv w_{\text{in}}$ and then

$$w_{1(s)} = w_{\text{in}} + \int_0^s \mathfrak{f}(s'; w_{\text{in}}, w_{\text{in}})\,\mathrm{d}s' \tag{II, 7}$$

respectively in our special case (II, 5)

$$w_{1(s)} = w_{\text{in}} + \int_0^s F_{(s')}\mathrm{d}s'\,. \tag{II, 7'}$$

Indeed, this is the neutral-particle-limit solution following from (II, 3) with $\Delta s \to 0$:

$$\dot{w}_{(s)} = F_{(s)}\,. \tag{II, 8}$$

The second approximation is assumed to be

$$w_{2(s)} = w_{\text{in}} + \int_0^s \mathfrak{f}(s'; w_{1(s')}, w_{1(s'+\Delta s)})\,\mathrm{d}s'\,, \tag{II, 9}$$

respectively

$$w_{2(s)} = w_{\text{in}} + \int_0^s F_{(s')}\,\mathrm{d}s'/\operatorname{Cosh}\Delta w_{1(s')}\,. \tag{II, 9'}$$

** The method of Peano [7, 8] is not applicable here, because this method would presume the knowledge of the exact value of $w_{(s)}$ in a distance Δs later than that point, in which the Newtonian initial data are given; whereas the Picard-Lindelöf method presupposes only an approximate value.

Continuing this procedure, a sequence of functions $w_{k(s)}$ is generated

$$w_{k(s)} = w_{\text{in}} + \int_0^s \mathfrak{f}(s'; w_{k-1(s')}, w_{k-1(s'+\Delta s)})\, ds', \quad \text{(II, 10)}$$

especially

$$w_{k(s)} = w_{\text{in}} + \int_0^s F_{(s')} ds' / \text{Cosh}\, \Delta w_{k-1(s')}\,. \quad \text{(II, 10')}$$

The initial condition

$$W_{(s=0)} = w_{\text{in}} \quad \text{(II, 11)}$$

is fulfilled for the limit function, because this is valid for all functions of the sequence $\{w_{k(s)}\}$.

Before the desired convergence proof (II, 6) for the sequence of functions (II, 10) is accomplished, some restrictions have to be imposed on the function $\mathfrak{f}(s; w_{(s)}, w_{(s+\Delta s)})$: Clearly, condition (II, 2a) will be extended to the present non-local case as

$$|\mathfrak{f}(s; w_1, w_2) - \mathfrak{f}(s; w_3, w_4)| \leqq N\{|w_1 - w_3| + |w_2 - w_4|\}\,; \quad \text{(II, 12)}$$

and (II, 2b) is generalized to

$$|\mathfrak{f}(s; w_1, w_2)| \leqq M\,, \quad \text{(II, 13)}$$

where one has for the special case (II, 5) (see Appendix)

$$2N = M = \max\{|F_{(s)}|\} =: F_{\max}\,. \quad \text{(II, 14)}$$

Clearly, M and N have to be finite (but may be arbitrarily large) constants.

These generalizations were rather evident, but in specifying the region $\mathfrak{G}(s, w)$ we have to distinguish two cases with respect to the range of the variable s (the "velocity" w is not restricted at all):

(α). The function $\mathfrak{f}(s; w_1, w_2)$ is either non-zero for all $s > 0$ and (w_1, w_2) arbitrary; or if $\mathfrak{f}(s; w_1, w_2)$ becomes zero for a certain value of s (s_0, say), then it is non-zero in some neighbouring point (s_1, say) so that $|s_0 - s_1| < \Delta s$. In this case (α), we can achieve a global convergence proof, provided the Lipschitz constant N satisfies the requirement

$$(2eN\Delta s) < 1\,; \quad e = 2{,}7182\ldots\,. \quad \text{(II, 15)}$$

As it shall be seen below, one is forced to consider the whole range of $s: 0 \leqq s < \infty$.

(β). The function $\mathfrak{f}(s; w_1, w_2)$ assumes the value zero in a whole interval of minimal length Δs:

$$\mathfrak{f}(s; w_1, w_2) = \begin{cases} 0 \text{ for (e.g.) } s_{\text{fin}} \leqq s \leqq s_{\text{fin}} + \sigma\,; \\ \qquad \sigma \geqq \Delta s \\ \text{such as in } (\alpha) \text{ for all other} \\ \text{values of } s \;\; (s > 0 \text{ assumed})\,. \end{cases} \quad \text{(II, 16)}$$

In this case (β) we can achieve an existence and uniqueness proof already for the interval $0 \leqq s \leqq s_{\text{fin}} + \sigma$ under the same condition as in (II, 15).

The reader shall easily realize what these statements on $\mathfrak{f}(s; w_1, w_2)$ mean for the force function $F_{(s)}$ in our special case (II, 5). The reason for discerning between the two cases (α) and (β) becomes clearer during the course of the convergence proof, which is readily attacked now.

Writing the limit function $W_{(s)}$ as

$$W_{(s)} = w_{0(s)} + \sum_{k=0}^{\infty} (w_{k+1(s)} - w_{k(s)})\,, \quad \text{(II, 17)}$$

we see that the uniform convergence of the sequence $\{w_{k(s)}\}$ is assured, if we can find a dominating series for the sum $\sum_{k=0}^{\infty} |w_{k+1(s)} - w_{k(s)}|$. Such a dominating series is constructed now by induction: First, we conclude from the Lipschitz condition (II, 12) in the usual way

$$\begin{aligned} |w_{k+1(s)} - w_{k(s)}| &\leqq \int_0^s ds' |\mathfrak{f}(s'; w_{k(s')}, w_{k(s'+\Delta s)}) \\ &\quad - \mathfrak{f}(s'; w_{k-1(s')}, w_{k-1(s'+\Delta s)})| \leqq N \int_0^s ds' \\ &\quad \cdot \{|w_{k(s')} - w_{k-1(s')}| + |w_{k(s'+\Delta s)} - w_{k-1(s'+\Delta s)}|\} \\ &\leqq 2N \int_0^{s+\Delta s} ds' |w_{k(s')} - w_{k-1(s')}|\,. \end{aligned} \quad \text{(II, 18)}$$

For $k = 0$ one finds

$$|w_{1(s)} - w_{0(s)}| \leqq \int_0^s |\mathfrak{f}(s'; w_{0(s')}, w_{0(s'+\Delta s)})|\, ds' \leqq Ms\,, \quad \text{(II, 19)}$$

or for $k = 1$

$$|w_{2(s)} - w_{1(s)}| \leqq 2N \tfrac{1}{2} M (s + \Delta s)^2\,, \quad \text{(II, 20)}$$

and for $k = 2$

$$|w_{3(s)} - w_{2(s)}| \leqq M(2N)^2 (s + 2\Delta s)^3/3! \text{ etc.} \quad \text{(II, 21)}$$

It is a matter of ease to establish the general formula

$$|w_{k+1(s)} - w_{k(s)}| \leqq \frac{M}{2N} \frac{[2N(s + k\Delta s)]^{k+1}}{(k+1)!} \quad \text{(II, 22)}$$

by induction on the basis of Eqs. (II, 18) up to (II, 21).

Before we make further progress in our proof, we stop for a moment at (II, 22) and observe that for $k \to \infty$ all future time ($s + k\Delta s \to \infty$) is involved on the right of this equation. This becomes understandable, if we look back at the recurrence formula (II, 10): in order to know $w_{k(s)}$ up to time s we must know $w_{k-1(s)}$ up to time $s+\Delta s$. But the knowledge of $w_{k-1(s+\Delta s)}$ requires the knowledge of w_{k-2} at time $s+2\Delta s$ etc. So for $k\to\infty$ we cannot restrict ourselves on a finite interval of proper time, but we must know $\mathfrak{f}(s; w_1, w_2)$ for all future values of s (especially: we must know all future forces $F_{(s)}$ in order to compute the "velocity" $w_{(s)}$ at a given finite time). In physical language, this is clearly a violation of causality. We do not want to go into detail now with respect to this point, which is left for future work, but we confine ourselves to point out that a certain "decoupling of the present behaviour from the future" is obtained in the above mentioned case (β), where no force is acting over a proper time interval of minimal length Δs. In this case, it is not necessary to know the future forces at times $s > s_{\text{fin}} + \sigma$, if we want to compute the "velocity" w at time s, where $0 \leqq s \leqq s_{\text{fin}}$. The reason for this is that $w_{(s)}$ is constant in the interval $s_{\text{fin}} \leqq s \leqq s_{\text{fin}} + \sigma$, and therefore we have to integrate in (II, 10) only up to $s = s_{\text{fin}}$ and shall then know also the value of $w_{k(s)}$ in the interval $s_{\text{fin}} \leqq s \leqq s_{\text{fin}} + \sigma$, where it is identical to $w_{k(s_{\text{fin}})}$.

After these remarks on the range of the variable s we continue the proof by making further evaluations concerning the inequality (II, 22):

$$|w_{k(s)} - w_{k-1(s)}| \leqq \frac{M}{2N} \frac{[2N\Delta s(n_s+k)]^k}{k!} \cdot \frac{n_s+k}{(k+1)} \cdot \frac{n_s+k}{(k+2)} \cdots \frac{n_s+k}{k+n_s}. \quad \text{(II, 23)}$$

Here we have substituted the right-hand neighbouring integer n_s ($n_s = 1, 2, 3, \ldots$; $n_s \geqq s/\Delta s$) for $s/\Delta s$, in order to handle only with integers, and further we have multiplied with n_s factors, each of which is greater than (or equals) unity. Hence, we have

$$|w_{k(s)} - w_{k-1(s)}| \leqq \frac{M}{2N} (2N\Delta s)^k \frac{(n_s+k)^{n_s+k}}{(k+n_s)!}. \quad \text{(II, 24)}$$

Now we can apply Stirling's formula

$$n! \geqq e(n/e)^n \quad \text{(II, 25)}$$

in the denominator and then obtain

$$|w_{k(s)} - w_{k-1(s)}| \leqq (M/2N)(2eN\Delta s)^k e^{n_s-1}. \quad \text{(II, 26)}$$

This means for the limit function (II, 17)

$$|W_{(s)}| \leqq |w_{0(s)}| + (M/2N)\, e^{n_s} \sum_{k=0}^{\infty} (2eN\Delta s)^k. \quad \text{(II, 27)}$$

But, by virtue of (II, 15), the sum converges and $|W_{(s)}|$ has $(M/2N)(1-2eN\Delta s)^{-1} e^{s/\Delta s}$ as a dominating series. This means that

1. $W_{(s)}$ is a finite number for all finite s,
2. $W_{(s)}$ satisfies the differential-difference Eq. (II, 4a) in the form

$$W_{(s)} = w_{\text{in}} + \int_0^s \mathfrak{f}(s'; W_{(s')}, W_{(s'+\Delta s)})\, ds',$$

because $\lim_{k\to\infty}$ and integration are interchangeable in the case of uniform convergence,

3. the initial condition $W_{(s=0)} = w_{\text{in}}$ is fulfilled.

Thus, we have actually constructed a solution of our problem.

A final remark has to be made on the restrictive condition (II, 15). From (II, 14) one concludes for the maximal force $K_{\max}$, which still guarantees the existence of solutions

$$e\, F_{\max}\, \Delta s < 1$$

$$K_{\max} < \frac{m c^2}{e \Delta s} = \frac{2}{3e} \frac{Z^2}{\Delta s^2} \approx \frac{Z^2}{(2\Delta s)^2} \quad \text{(II, 28)}$$

i. e. the maximal force $K_{\max}$ occurs, if the centers of two electrons are separated by two times the classical radius; this means that the two electrons come into contact. For such "contact forces" our proof fails, but it is clear that such strong forces are of no experimental relevance in the classical domain. However, one cannot conclude that the existence of solutions is forbidden in the case $K > K_{\max}$. Indeed, we shall present below an exact solution of the constant-force problem, which is not restricted to $K < K_{\max}$. This suggests that there may be an existence proof working without the restriction (II, 15).

III. Uniqueness of the Solution

We wish now to prove that the solution $W_{(s)}$ found by the method of successive approximations is the only possible solution of the initial-value problem under consideration. To this end, we assume that there be a second solution $v_{(s)}$ ($\neq W_{(s)}$) and shall then show, that $v_{(s)}$ must be identical to $W_{(s)}$.

The deviation of the new solution $v_{(s)}$ from one of the approximate solutions $w_{k(s)}$ is

$$|v_{(s)} - w_{k(s)}| \leqq \int_0^s |\mathfrak{f}(s'; v_{(s')}, v_{(s'+\Delta s)}) - \mathfrak{f}(s'; w_{k-1(s')}, w_{k-1(s'+\Delta s)})|\, ds' \quad \text{(III, 1)}$$

and therefore by virtue of the Lipschitz condition (II, 12)

$$|v_{(s)} - w_{k(s)}| \leqq N \int_0^s ds' \{|v_{(s')} - w_{k-1(s')}| + |v_{(s'+\Delta s)} - w_{k-1(s'+\Delta s)}|\} \leqq 2N \int_0^{s+\Delta s} ds' |v_{(s')} - w_{k-1(s')}| . \quad \text{(III, 2)}$$

The lowest-order deviation is here

$$|v_{(s)} - w_{0(s)}| \leqq \int_0^s |\mathfrak{f}(s'; v_{(s')}, v_{(s'+\Delta s)})| ds' \leqq M s . \quad \text{(III, 3)}$$

Next, one finds with the help of (III, 2)

$$|v_{(s)} - w_{1(s)}| \leqq 2 N M \tfrac{1}{2}(s+\Delta s)^2 , \quad \text{(III, 4)}$$

or as well

$$|v_{(s)} - w_{2(s)}| \leqq M(2N)^2 \frac{1}{3!} (s + 2\Delta s)^3 \text{ etc.} \quad \text{(III, 5)}$$

From here one obtains easily the general deviation formula by means of induction

$$|v_{(s)} - w_{k(s)}| \leqq M(2N)^k \frac{[s + k\Delta s]^{k+1}}{(k+1)!} . \quad \text{(III, 6)}$$

Using quite similar evaluations as in the preceding section, the final result is

$$|v_{(s)} - w_{k-1(s)}| \leqq (M/2N)(2eN\Delta s)^k e^{s/\Delta s} . \quad \text{(III, 7)}$$

So we see that for all finite s the deviation tends to zero in the limit $k \to \infty$, if condition (II, 15) is satisfied

$$\lim_{k\to\infty} |v_{(s)} - w_{k(s)}| = 0 \quad \text{q.e.d.} \quad \text{(III, 8)}$$

Hence, $v_{(s)}$ must be identical with the limit function $W_{(s)}$, and the latter one is the only solution of the initial-value problem under consideration.

IV. An Exact Solution: the Short Pulse

Assume now the reduced force $F_{(s)}$ of the equation of motion (II, 3) to be non-zero only during a proper time interval of maximal length Δs. If s is contained in this interval, $w_{(s+\Delta s)}$ must be constant, because $\dot{w}_{(s+\Delta s)}$ is zero on account of the equation of motion. Hence, we can write

$$-\dot{w}_{(s)} \operatorname{Cosh}[\Delta w_{(s)}] = \left(\frac{d}{ds} \Delta w_{(s)}\right) \cdot \operatorname{Cosh}[\Delta w_{(s)}] = -F_{(s)} \quad \text{(IV, 1)}$$

or

$$\frac{d}{ds} \operatorname{Sinh} \Delta w_{(s)} = -F_{(s)} , \quad \text{(IV, 2)}$$

which can be easily integrated as

$$\Delta w_{(s)} = \operatorname{Ar\,Sinh} \{\int_s^{s_{\text{fin}}} F_{(s')} ds'\} . \quad \text{(IV, 3)}$$

Hereby, s_{fin} is that final time at which $F_{(s)}$ becomes zero: $F_{(s)} \equiv 0$ for $s > s_{\text{fin}}$. By means of differentiation

$$\dot{w}_{(s)} = \frac{F_{(s)}}{\sqrt{1 + \{\int_s^{s_{\text{fin}}} F_{(s')} ds'\}^2}} \quad \text{(IV, 4)}$$

and substitution back into the equation of motion (II, 3), it is verified readily that (IV, 3) is indeed the unique solution of (II, 3). Denoting the constant final "velocity" w for $s \geqq s_{\text{fin}}$ by w_{fin}, one finds from (IV, 3)

$$w_{(s)} = w_{\text{fin}} - \operatorname{Ar\,Sinh}[\int_s^{s_{\text{fin}}} F_{(s')} ds'] , \quad \text{(IV, 5)}$$

and the total increment in "velocity" w is

$$w_{\text{fin}} - w_{\text{in}} = \operatorname{Ar\,Sinh}[\int_{s_{\text{in}}}^{s_{\text{fin}}} F_{(s')} ds'] , \quad \text{(IV, 6)}$$

where s_{in} is that time, where $F_{(s)}$ becomes non-zero: $F_{(s)} \equiv 0$ for $s \leqq s_{\text{in}}$. So the length of the interval under consideration is $|s_{\text{fin}} - s_{\text{in}}|$ and we must have therefore

$$|s_{\text{fin}} - s_{\text{in}}| \leqq \Delta s . \quad \text{(IV, 7)}$$

It must be stressed that (IV, 5) is an exact solution, no means whether $F_{(s)}$ is bounded or not [cf. Equation (II, 28)]. This suggests that there is a unique solution of our initial-value problem even in the general case, where the restrictive condition (II, 15) is violated. The only requirement left would then consist in an integrability condition on the force $F_{(s)}$.

Comparing now formula (IV, 4) with the corresponding one following from the integro-differential formulation (I, 2) of the Lorentz-Dirac theory

$$\dot{w}_{(s)} = \int_s^{\infty} F_{(s')} \exp\{-(s'-s)/\Delta s\} ds'/\Delta s , \quad \text{(IV, 8)}$$

one realizes readily that also in the present nonlocal theory the acceleration at time s is determined by forces acting on the particle at later times. Thus, there must be violation of causality even in the present model. But this violation is presumably less severe than in the Lorentz-Dirac case.

Of course, there is an intimate connection between the possibility of imposing Newtonian initial data and causality-violating phenomena: The solution of the equation of motion can be subjected to that sort of initial conditions, because the rate of change of the "velocity" w is determined by w itself at later times; or if w is eliminated, by the forces at later times [see Eq. (IV, 4)]. Thus, the pre-history of the particle is irrelevant in this respect.

The force $F_{(s)}$ in the foregoing formulae is now assumed to be a constant (f_c, say), so that

$$F_{(s)} = \begin{cases} 0; & s < s_{\text{in}} \\ f_c; & s_{\text{in}} \leqq s < s_{\text{fin}} \\ 0; & s_{\text{fin}} \leqq s\,. \end{cases} \tag{IV, 9}$$

In the Lorentz-Dirac case we get the well-known solution (Rohrlich, loc. cit., p. 178)

$$\dot w^{\text{LD}}{}_{(s)} = \begin{cases} f_c \exp\{s/\Delta s\}(1 - \exp\{-l_c/\Delta s\})\,; & s \leqq s_{\text{in}} = 0 \\ f_c\,(1 - \exp\{-(l_c - s)/\Delta s\})\,; & 0 \leqq s \leqq s_{\text{fin}} \equiv l_c \\ 0\,; & l_c \leqq s \end{cases} \tag{IV, 10}$$

which clearly exhibits the effect of pre-acceleration * (see Fig. 1).

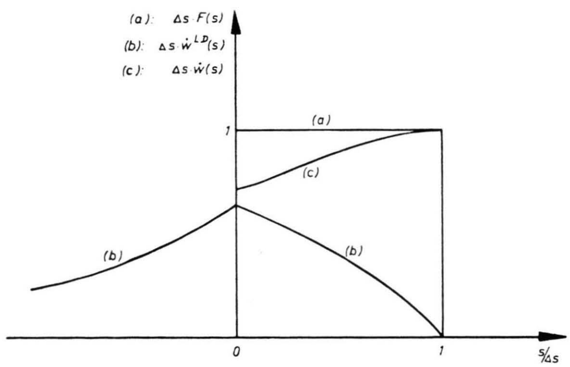

Fig. 1. External force $F_{(s)}$, Lorentz-Dirac acceleration $\dot w^{\text{LD}}{}_{(s)}$, and the acceleration $\dot w_{(s)}$, following from the present theory, are plotted in reduced units for the special force field (IV, 9). We have chosen $l_c = \Delta s$; $f_c\,\Delta s = 1$. Observe the qualitatively quite distinct behaviour of the accelerations!

On the other hand, Eq. (IV, 4) yields

$$\dot w_{(s)} = \begin{cases} 0\,; & s < 0 \\ f_c/\sqrt{1 + f_c^2 (l_c - s)^2}\,; & 0 \leqq s < l_c \\ 0\,; & l_c \leqq s\,. \end{cases} \tag{IV, 11}$$

* Pre-acceleration is understood here in the usual sense: the particle accelerates, before the force is switched on.

This function is also plotted in Fig. 1, and the reader should realize clearly that the unphysical effects mentioned usually in connection with this special force problem [2, 9] are missing in the present theory.

A final remark must be made about what Rohrlich (loc. cit., p. 213) has called the "neutral-particle limit". He has observed that if one lets Δs in (IV, 10) tend to zero, then the acceleration $\dot w^{\text{LD}}{}_{(s)}$ reduces to that of a neutral particle in the same force field, namely

$$\dot w^{\text{np}}{}_{(s)} = \begin{cases} 0\,; & s < 0 \\ f_c\,; & 0 \leqq s < l_c \\ 0\,; & l_c \leqq s\,. \end{cases} \tag{IV, 12}$$

But Ingraham [9] has pointed out recently, that the solution (IV, 10) is not analytic in the variable Δs (understood in the complex Δs-plane) and can therefore not be expanded in a perturbation series with respect to Δs, whereby the lowest order approximation would be the neutral-particle limit (IV, 12). With this difficulty of the Lorentz-Dirac solution we have not to bother in the present case (IV, 11), because this solution does not depend upon Δs at all. This is due to the fact that the pulse duration is shorter than Δs. If the force duration is longer than Δs, the possibility of a perturbative treatment with respect to Δs is suggested by the considerations of the next section.

V. Successive Approximation for the Constant-force Problem

In this section we look for an approximate solution for the force problem given in (IV, 9). According to the Picard-Lindelöf method, one starts with the neutral particle limit (IV, 12)

$$\dot w_{1(s)} \equiv \dot w^{\text{np}}{}_{(s)}\,, \tag{V, 1}$$

but with l_c ($\leqq \Delta s$) replaced by L ($\geqq \Delta s$ assumed). Thus

$$w_{1(s)} = \begin{cases} w_{\text{in}}\,; & -\infty < s \leqq 0 \\ w_{\text{in}} + f_c \cdot s\,; & 0 \leqq s \leqq L \\ w_{\text{fin}}\,; & L \leqq s\,. \end{cases} \tag{V, 2}$$

Clearly, w_{in} and w_{fin} stand for initial and final "velocity"; especially we have in the neutral particle limit

$$w_{\text{fin}} = w_{\text{in}} + f_c \cdot L\,. \tag{V, 3}$$

From (II, 9′) one finds for the next approximation

$$w_{2(s)} = \begin{cases} w_{\text{in}}\,; \quad s \leqq 0 \\ w_{\text{in}} + f_{\text{c}} \cdot \int\limits_0^s \dfrac{\mathrm{d}s'}{\operatorname{Cosh}(f_{\text{c}} \Delta s)} = w_{\text{in}} + \dfrac{f_{\text{c}} \cdot s}{\operatorname{Cosh}(f_{\text{c}} \Delta s)}\,; \quad 0 \leqq s \leqq L - \Delta s \\ w_{\text{in}} + f_{\text{c}} \cdot \dfrac{L - \Delta s}{\operatorname{Cosh}(f_{\text{c}} \Delta s)} + f_{\text{c}} \cdot \int\limits_{s' = L - \Delta s}^{s} \dfrac{\mathrm{d}s'}{\operatorname{Cosh}[f_{\text{c}}(L - s')]}\,; \quad L - \Delta s \leqq s \leqq L \\ w_{\text{fin}}\,; \quad L \leqq s\,. \end{cases} \tag{V, 4}$$

In the middle lines of (V, 4), we recognize the radiation damping effect in the replacement of f_{c} in (V, 2) by $f_{\text{c}}/\operatorname{Cosh}(f_{\text{c}} \Delta s)$, and the third line of (V, 4) exhibits the non-local effect, which produces a sort of rounding-off. We cannot expect that this rounding-off is correctly accounted for by the third line of (V, 4). Rather, one would assume that in this second approximation (V, 4) the exact solution $W_{(s)}$ is approximated correctly up to order f_{c}^3 [or $(\Delta s\, f_{\text{c}})^3$ in dimensionless units]. Note, that the neutral particle limit (V, 2) contains the force f_{c} linearly. Accordingly, we can expand ** the Cosh-function in (V, 4) to obtain

$$w_{2(s)} \approx w_{\text{in}} + f_{\text{c}}\, s \cdot (1 - \tfrac{1}{2} f_{\text{c}}^2 \Delta s^2)\,; \quad 0 \leqq s \leqq L - \Delta s \tag{V, 4a}$$

and

$$w_{2(s)} \approx w_{\text{in}} + f_{\text{c}}(L - \Delta s) \cdot (1 - \tfrac{1}{2} f_{\text{c}}^2 \Delta s^2) + f_{\text{c}} \cdot \int\limits_{s' = L - \Delta s}^{s} \mathrm{d}s' \{1 - \tfrac{1}{2} f_{\text{c}}^2 (L - s')^2\}\,; \quad L - \Delta s \leqq s \leqq L\,. \tag{V, 4b}$$

After a trivial integration one gets

$$w_{2(s)} \approx w_{\text{in}} + f_{\text{c}} \cdot s + \frac{1}{3} (f_{\text{c}} \Delta s)^3 \left\{1 - \frac{3}{2} \frac{L}{\Delta s}\right\} + \frac{1}{6} f_{\text{c}}^3 (L - s)^3\,; \quad L - \Delta s \leqq s \leqq L\,. \tag{V, 4b′}$$

Hence, the lowest order corrections of the final "velocity" (V, 3) are

$$w_{2,\,\text{fin}} = w_{2(\text{L})} \approx w_{\text{in}} + f_{\text{c}} L + \frac{1}{3} (f_{\text{c}} \Delta s)^3 \left\{1 - \frac{3}{2} \frac{L}{\Delta s}\right\}, \tag{V, 5}$$

and therefore we can rewrite Eq. (V, 4b′) as

$$w_{2(s)} \approx w_{2,\,\text{fin}} - f_{\text{c}}(L - s) + \tfrac{1}{6} f_{\text{c}}^3 (L - s)^3\,. \tag{V, 4b″}$$

** This procedure is in close analogy to the approximation proposed in Ref. [4], where one finds as an approximation $\dot{w}(1 + \frac{1}{2} \Delta w^2) \approx \dot{w}(1 + \frac{1}{2} \Delta s^2 \dot{w}^2) \approx F_{(s)}$ in place of the Lorentz-Dirac equation.

But in this form, we can easily check the consistency of this sort of approximation by simply putting $L = \Delta s = l_{\text{c}}$ and comparing the resulting formulae (V; 5, 4b″) with those following from (IV, 5) and (IV, 6), specializing $F_{(s)}$ to the force (IV, 9): the terms of first and third power in $(L - s)$ represent the lowest order terms of the Taylor series expansion of the Ar Sinh-function in (IV; 5, 6). Clearly, formulae (IV; 5, 6) are exact for the last Δs-interval of an arbitrary force. Figure 2 shows a comparison of the accelerations $\dot{w}^{\text{np}}{}_{(s)}$, $\dot{w}^{\text{LD}}{}_{(s)}$ and $\dot{w}_{2(s)}$.

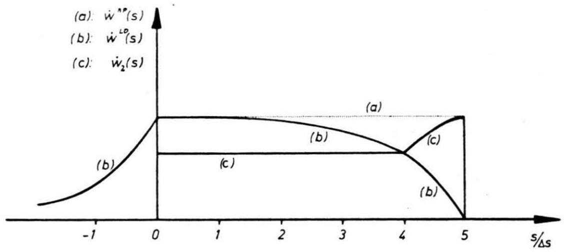

Fig. 2. The second approximation $\dot{w}_{2(s)}$ is compared with the Lorentz-Dirac solution $\dot{w}^{\text{LD}}{}_{(s)}$ and the neutral particle case $\dot{w}^{\text{np}}{}_{(s)}$. The latter is identical as well to $\dot{w}_{1(s)}$ as to $F_{(s)}$. The force duration is $L = 5\, \Delta s$ and the force strength $f_{\text{c}} \cdot \Delta s = 1$.

Looking back at the expansions in Eqs. (V, 4), one realizes readily that this sort of approximation procedure results actually in a power series expansion of the exact solution $W_{(s)}$ with respect to the extension parameter Δs [or $(f\, \Delta s)$ in reduced units]. Though a general proof of the property of analyticity of the exact solution $W_{(s)}$ with respect to the variable Δs is an outstanding problem, the approximation procedure indicated above suggests this analyticity property. The correctness of the latter assumed, we can put our aversion against the Lorentz-Dirac theory in a more concrete form: Since the Lorentz-Dirac solution (IV, 8) contains a function [namely $\exp\{-(s' - s)/\Delta s\}$], which has an essential singularity in the (complex) variable Δs

$= \frac{2}{3} Z^2/m c^2$ at $\Delta s = 0$, this solution is not expansible with respect to Δs in the general case * (see constant-force problem). Hence, it cannot be considered as an approximation of the present finite-size theory being analytical in Δs presumably. This statement sets aside the earlier opinion [11] that at least the integro-differential formulation of the Lorentz-Dirac theory be useful.

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Appendix

The Special Function $\mathfrak{f}(s; \Delta w) = F_{(s)}/\text{Cosh}\, \Delta w$

In order to show that conditions (II, 12) to (II, 14) can be satisfied for the special function $\mathfrak{f}(s; \Delta w)$ from (II, 5), one calculates first

$$|\mathfrak{f}(s; \Delta w_2) - \mathfrak{f}(s; \Delta w_1)| = |F_{(s)}| \cdot \left| \frac{1}{\text{Cosh}\, \Delta w_2} - \frac{1}{\text{Cosh}\, \Delta w_1} \right| . \quad (\text{A}, 1)$$

Then one verifies

$$\left| \frac{1}{\text{Cosh}\, \Delta w_2} - \frac{1}{\text{Cosh}\, \Delta w_1} \right| \leqq \tfrac{1}{2} |\Delta w_1 - \Delta w_2| , \quad (\text{A}, 2)$$

by looking for the maximal slope, which a straight line, intersecting the curve $1/\text{Cosh}\, \Phi$ at least twice,

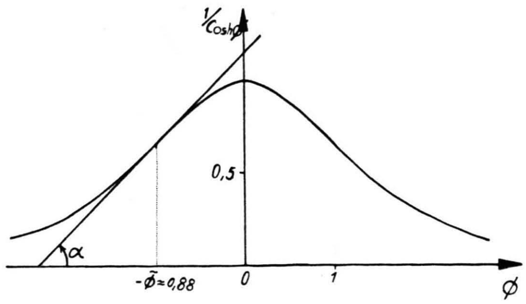

Fig. A.

can assume. The figure A shows such a straight line in its extremal position: it is the tangent to the curve in a point where the curvature changes sign, and simultaneously it is the tangent with the steepest slope. Therefore, $\tilde{\Phi}$ is the solution of the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\tilde{\Phi}^2} \left(\frac{1}{\text{Cosh}\, \tilde{\Phi}} \right) = \frac{\text{Sinh}\, \tilde{\Phi} - 1}{\text{Cosh}^3\, \tilde{\Phi}} = 0 . \quad (\text{A}, 3)$$

Hence ($\Phi \longleftrightarrow \Delta w$)

$$|\tan \alpha|_{\max} = \left| \frac{\frac{1}{\text{Cosh}\, \Delta w_2} - \frac{1}{\text{Cosh}\, \Delta w_1}}{\Delta w_2 - \Delta w_1} \right|_{\max} = \left| \frac{\mathrm{d}}{\mathrm{d}\tilde{\Phi}} \left(\frac{1}{\text{Cosh}\, \tilde{\Phi}} \right) \right| = \left| \frac{\text{Sinh}\, \tilde{\Phi}}{1 + \text{Sinh}^2\, \tilde{\Phi}} \right| = \frac{1}{2} . \quad (\text{A}, 4)$$

So, conditions (II, 12) and (II, 13) are satisfied (put $w_1 - w_2 = \Delta w_1$; $w_3 - w_4 = \Delta w_2$). As for condition (II, 14), we have with the above results (A, 1) and (A, 4)

$$2 N = M = |F_{(s)}|_{\max} . \quad (\text{A}, 5)$$

* This opinion stands in contrast to that of Bhabha [10] and Rohrlich (loc. cit., p. 214).

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